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Descent and Essential Descent Spectrum of Linear Relations

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Abstract: In this paper, we study the descent spectrum and the essential descent spectrum of linear relations everywhere defined on Banach spaces. We prove that the corresponding spectra are closed and we obtain that a Banach space X is finite dimensional if and only if the descent and the essential descent of every closed linear relation acting in X is finite. We give characterizations of the descent and the essential descent of linear relations and as applications, some perturbation results are presented.

Key words: Descent, essential descent, spectrum, linear relations.

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1. Introduction

The notion of descent and essential descent of linear operators was studied in several articles, for instance, we cite [5],[6], [11], [13] and [22]. In [3], [10] and [18], this concept is extended to the case of linear relations. Many properties of descent for the case of linear operators remain be valid in the context of linear relations, sometimes under supplementary conditions. M. Burgos, A. Kaidi, M. Mbekhta and M. Oudghiri in [6] and O. Bel Hadj Fredj in [5] studied the descent spectrum and the essential descent spectrum of an operator acting in Banach spaces. They show that the corresponding spectra are compact subsets of the spectrum, and that for T a bounded operator in X, $\sigma_{des}(T)$ (respectively $\sigma^e_{des}(T)$) is empty precisely when T is algebraic. Furthermore, they establish that the descent (respectively essential descent) of every operator acting in X is finite if and only if X has finite dimension. The purpose of this paper is to extend the results of the type mentioned above to multivalued linear relations in Banach spaces.

To make the paper easily accessible some results from the theory of linear relations in normed spaces due to R. W. Cross [9] and A. Sandovici, H. de Snoo,

H. Winkler [18] are recalled in Section 2. In particular, results concerning the descent of a linear relation are presented. Section 3 is devoted to the study of the descent spectrum and the essential descent spectrum of a closed linear relation T everywhere defined in a Banach space X. We show that the corresponding spectra are two closed subsets of $\sigma(T)$, and that they are empty precisely when the essential descent resolvent contains the boundary of the spectrum of T.

Throughout this paper, X will be, unless otherwise stated, an infinite dimensional complex Banach space. A multivalued linear operator in X or simply a linear relation in X, $T : X \to X$ is a mapping from a subspace $D(T) \subset X$, called the domain of T, into the collection of nonempty subsets of X such that $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T x_1 + \alpha_2 T x_2$ for all nonzero α_1, α_2 scalars and $x_1, x_2 \in D(T)$. We denote the class of linear relations in X by $LR(X)$. If T maps the points of its domain to singletons, then T is said to be single-valued or simply operator. A linear relation $T \in LR(X)$ is uniquely determined by its graph, G(T), which is defined by $G(T) := \{(x, y) \in X \times X : x \in D(T) \text{ and } y \in Tx\}$. We say that T is closed if its graph is a closed subspace of $X \times X$, furthermore the class of all closed linear relations in X will be denoted by $CR(X)$. The inverse of T is the linear relation T^{-1} given by $G(T^{-1}) := \{(y, x) : (x, y) \in G(T)\}$. The subspace $T^{-1}(0)$, denoted by $N(T)$, is called the null space of T and we say that T is injective if $N(T) = \{0\}$. The range of T is the subspace $R(T) := T(X)$ and T is said to be surjective if its range coincides with X. We have the following identities

$$D(T^{-1}) = R(T), \quad R(T^{-1}) = D(T), \quad N(T^{-1}) = T(0).$$

Furthermore, we define the nullity and the defect of T by

$$\alpha(T) := \dim(N(T)) \ \text{ and } \ \beta(T) := \operatorname{codim}(R(T)),$$

respectively.

Let M be a subspace of X such that $M \cap D(T) \neq \emptyset$. Then the restriction of T to M, denoted by $T_{/M}$, is given by $G(T_{/M}) := \{(m, y) \in G(T) : m \in M \cap D(T)\}$. For linear relations S and T such that $D(T) \cap D(S) \neq \emptyset$, the sum $S + T$ is the linear relation given by $G(S + T) := \{(x, y + z) : (x, y) \in G(S), (x, z) \in G(T)\}$ and the relation λT, for $\lambda \in \mathbb{C}$, is defined by $G(\lambda T) := \{(x, \lambda y) : (x, y) \in G(T)\}$. While $T - \lambda$ stands for $T - \lambda I$, where I is the identity operator on X. Let $S, T \in LR(X)$, the product ST is defined as the relation $G(ST) := \{(x, z) : (x, y) \in G(T), (y, z) \in G(S) \text{ for some } y \in X\}$. The product of relations is clearly associative. Hence for $T \in LR(X)$ and

$n \in \mathbb{Z}$, T^n is defined as usual with $T^0 = I$ and $T^1 = T$. It is easily seen that $(T^{-1})^n = (T^n)^{-1}, n \in \mathbb{Z}$.

We define the singular chain manifold of $T \in LR(X)$ (see [18, (3.3)]) by

$$R_c(T) := \left(\bigcup_{n=0}^{+\infty} N(T^n) \right) \cap \left(\bigcup_{n=0}^{+\infty} T^n(0) \right).$$

We say that T has trivial singular chain if $R_c(T) = \{0\}$. Let M and N be subspaces of X and X' (the dual space of X) respectively. Then $M^\perp := \{x' \in X' : x'(M) = 0\}$ and $N^\top := \{x \in X : x'x = 0, \forall x' \in N\}$. The adjoint T' of T is defined by

$$G(T') := G(-T^{-1})^\perp \subset X' \times X'.$$

We let $Q_T : X \to X/\overline{T(0)}$ be the natural quotient map on X with kernel $\overline{T(0)}$. Clearly $Q_T T$ is a linear operator and that $T \in LR(X)$ is closed if and only if $Q_T T$ is closed and $T(0)$ is closed in $R(T)$ (see [9, II.5.3]). For $x \in X$ we define $\|Tx\| := \|Q_T Tx\|$ and thus $\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \|Q_T T\|$. We note that this quantity is not a true norm since $\|T\| = 0$ does not imply $T = 0$. T is said to be continuous if for each open set V in $R(T)$, $T^{-1}(V)$ is an open set in $D(T)$ equivalently $\|T\| < \infty$, open if its inverse is continuous equivalently $\gamma(T) > 0$, where $\gamma(T) := \sup\{\lambda \geq 0 : \lambda d(x, N(T)) \leq \|Tx\|, x \in D(T)\}$. Continuous everywhere defined linear relations are referred to as bounded relations. Let $T \in CR(X)$, we say that T is lower semi-Fredholm, if $R(T)$ is closed and $\operatorname{codim} R(T)$ is finite.

For $T \in LR(X)$, the kernels and the ranges of the iterates $T^n, n \in \mathbb{N}$, form two increasing and decreasing chains, respectively, i.e the chain of kernels

$$N(T^0) = \{0\} \subset N(T) \subset N(T^2) \subset \cdots,$$

and the chain of ranges

$$R(T^0) = X \supset R(T) \supset R(T^2) \supset \cdots.$$

Furthermore, (see [18, Lemma 3.4]), if $N(T^k) = N(T^{k+1})$ for some $k \in \mathbb{N}$, then $N(T^n) = N(T^k)$ for all nonnegative integers $n \geq k$ and if $R(T^k) = R(T^{k+1})$ for some $k \in \mathbb{N}$, then $R(T^n) = R(T^k)$ for all nonnegative integers $n \geq k$. This statement leads to define the ascent of T by

$$a(T) := \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\},$$

whenever these minima exist. If no such numbers exist the ascent of T is defined to be ∞. Likewise, this statement leads to the introduction of the descent of T by

$$d(T) := \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\},$$

the infimum over the empty set is taken to be ∞. For $T \in LR(X)$, we consider the decreasing sequence $\beta_n(T) := \dim R(T^n)/R(T^{n+1}), n \in \mathbb{N}$ (see Lemma 2.3 below). We shall say that T has finite essential descent if

$$d_e(T) := \inf\{n \in \mathbb{N} : \beta_n(T) < \infty\},$$

where the infimum over the empty set is taken to be infinite, is finite. Clearly $d(T) = \inf\{n \in \mathbb{N} : \beta_n(T) = 0\}$ and if $T \in CR(X)$ is lower semi-Fredholm then T has finite essential descent precisely we have $d_e(T) = 0$. Evidently, a linear relation with finite descent has a finite essential descent. In the case when $d_e(T) < \infty$ we denote $q(T) := \inf\{q \in \mathbb{N} :\ \beta_n(T) = \beta_q(T),\ \forall n \geq q\}$.

Let $T \in CR(X)$. The resolvent set of T is the set

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is injective and surjective}\}.$$

The spectrum of T is the set $\sigma(T) : \mathbb{C}\backslash\rho(T)$. Recall (see [9, VI.1.3]), that $\rho(T)$ is an open set and hence $\sigma(T)$ is a closed subset of $\mathbb{C}$. The ascent resolvent, the descent resolvent and the essential descent resolvent sets of a linear relation $T \in LR(X)$ are respectively defined by

$$\rho_{asc}(T) := \{\lambda \in \mathbb{C} : a(T-\lambda) < \infty \text{ and } R((T-\lambda)^{a(T-\lambda)+1}) \text{ is closed}\},$$
$$\rho_{des}(T) := \{\lambda \in \mathbb{C} : d(T-\lambda) < \infty\},$$
$$\rho^e_{des}(T) := \{\lambda \in \mathbb{C} : d_e(T-\lambda) < \infty\}.$$

The complementary sets $\sigma_{asc}(T) := \mathbb{C}\backslash\rho_{asc}(T)$, $\sigma_{des}(T) := \mathbb{C}\backslash\rho_{des}(T)$ and $\sigma^e_{des}(T) := \mathbb{C}\backslash\rho^e_{des}(T)$ are the ascent spectrum, the descent spectrum and the essential descent spectrum of T, respectively.

2. Preliminary and auxiliary results

In this section we collect some results of the theory of multivalued linear operators which are used to prove the main results in Section 3. The proof of the next lemma can be found in [9].

Lemma 2.1. ([9, I.3.1]) *Let $T \in LR(X,Y)$. Then*

(i) $TT^{-1}(M) = M \cap R(T) + T(0)$, *for all $M \subset Y$,*

(ii) $T^{-1}T(M) = M \cap D(T) + N(T)$, *for all $M \subset X$,*

(iii) $T(M+N) = T(M) + T(N)$, *for all $M \subset X$ and $N \subset D(T)$.*

The next lemma is elementary but it is essential to prove Lemma 2.3.

Lemma 2.2. ([18, Lemma 2.1 and Lemma 2.3]) *Let M and N be subspaces of a vector space X. Then*

(i) $\dim M/M \cap N = \dim(M+N)/N$.

(ii) *Assume that $N \subset M$ then* $\dim X/N = \dim X/M + \dim M/N$.

Let T be a linear relation acting on X. The sequence $\beta_n(T) := \dim R(T^n)/R(T^{n+1})$ play the fundamental role in the definition of the essential descent of T. In the next lemma we show that this sequence is decreasing.

Lemma 2.3. *Let $T \in LR(X)$. Then*

(i) $\dim R(T^n)/R(T^{n+1}) \leq \dim R(T^{n-1})/R(T^n)$ *for all $n \geq 1$.*

(ii) *If there exists $n \in \mathbb{N}$ such that $\dim R(T^n)/R(T^{n+1})$ is finite, then $\dim R(T^m)/R(T^{m+1})$ is finite for all $m \geq n$.*

(iii) $\dim R(T^n)/R(T^{n+1}) < \infty$ *if and only if* $\dim R(T^n)/R(T^{n+k}) < \infty$ *for all $k \geq 1$.*

Proof. (i) Let $y_1, y_2, \ldots, y_k \in R(T^n)$ such that $\overline{y_1}, \overline{y_2}, \ldots, \overline{y_k}$ are linearly independent in $M_{n+1} = R(T^n)/R(T^{n+1})$. Then there exist $x_1, x_2, \ldots, x_k \in R(T^{n-1})$ such that $y_i \in Tx_i$ for all $1 \leq i \leq k$. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be scalars such that

$$\overline{0} = \sum_{i=1}^{k} \alpha_i \overline{x_i} = \overline{\sum_{i=1}^{k} \alpha_i x_i} \text{ in } M_n.$$

Then $\sum_{i=1}^{k}\alpha_i x_i \in R(T^n)$. It follows that $\sum_{i=1}^{k}\alpha_i y_i \in R(T^{n+1})$, and hence $\sum_{i=1}^{k}\alpha_i\overline{y_i} = \overline{0}$. Thus $\alpha_i = 0$ for all $1 \leq i \leq k$, which implies that $\overline{x_1}, \overline{x_2}, \dots \overline{x_k}$ are linearly independent in M_n. Now for any k linearly independent vectors in M_{n+1} there exist k linearly independent vectors in M_n. This implies that $\dim M_{n+1} \leq \dim M_n$.

(ii) By induction, suppose $\dim R(T^m)/R(T^{m+1}) < \infty$ for some $m \geq n$. Then, by (i), $\dim R(T^{m+1})/R(T^{m+2}) \leq \dim R(T^m)/R(T^{m+1}) < \infty$.

(iii) Assume that $\dim R(T^n)/R(T^{n+1}) < \infty$ then by Lemma 2.2,

$$\dim R(T^n)/R(T^{n+k}) = \sum_{i=0}^{k-1} \dim R(T^{n+i})/R(T^{n+i+1}) < \infty.$$

∎

We shall make frequent use of the following result which is the multivalued version of the corresponding result for operators. It is used essentially to prove Theorem 3.1.

Lemma 2.4. ([18, Lemma 4.1 and Lemma 4.4]) *Let $T \in LR(X)$ be everywhere defined and let $n, m \in \mathbb{N}$. Then*

(i) *$R(T^m)/R(T^{m+n}) \cong X/R(T^n) + N(T^m)$.*
(ii) *If, moreover, $R_c(T) = \{0\}$, then $N(T^{m+n})/N(T^n) \cong N(T^m) \cap R(T^n)$.*

There exist closed linear relations S and T such that $S + T$ is not closed. We shall use the following result which gives sufficient conditions for the sum of two closed linear relations to be closed.

Lemma 2.5. ([4, Lemma 14]) *Let X be a Banach space and let $S, T \in CR(X)$ be continuous with $S(0) \subset T(0)$ and $D(T) \subset D(S)$. Then $S + T$ is closed.*

The proof of the next result can be found in [18].

Lemma 2.6. ([18, Lemma 7.2]) *Let $T \in LR(X)$. Then $N(T-\lambda)^n \subset R(T^m)$ for all $n, m \in \mathbb{N}$ and for each $\lambda \neq 0$.*

An important role is played by certain root manifolds of a relation T in X. We have the following results which are sometimes useful.

Lemma 2.7. *Let $T \in LR(X)$. Then*

(i) *If $R_c(T) = \{0\}$, then $R_c(T_{/M}) = \{0\}$ for all subspaces M of X.*

(ii) *$R_c(T) = \{0\}$ if and only if $R_c(T-\lambda) = \{0\}$, for all $\lambda \in \mathbb{C}$.*

(iii) *$\rho(T) \neq \emptyset$ if and only if $\rho(T-\lambda) \neq \emptyset$, for all $\lambda \in \mathbb{C}$.*

(iv) *If $\rho(T) \neq \emptyset$, then $R_c(T) = \{0\}$.*

Proof. (i) and (ii) are proved in [20, Lemma 3.1 and Lemma 7.1]. (iii) is trivial and (iv) is a direct consequence of [18, Lemma 6.1] . ∎

It is very well known that for a bounded operator in X, $T^{-1}(M)$ is closed whenever M is a closed subspace of X. In the following we give conditions for $T^{-1}(M)$ to be closed if T is a multivalued linear operator.

Lemma 2.8. *Let $T \in CR(X)$ be everywhere defined and let M be a closed subspace of X such that $T(0) \subset M$. Then $T^{-1}(M)$ is closed.*

Proof. Since T is closed and everywhere defined then T is bounded (see [9, III.4.2]), and hence $Q_T T$ is a bounded operator. On the other hand $Q_T(M) = (M + T(0))/T(0) = M/T(0)$ is closed (as M and $T(0)$ are both closed, see [4, Lemma 13]). Hence $(Q_T T)^{-1} Q_T(M)$ is closed. But trivially we have that $(Q_T T)^{-1} Q_T(M) = T^{-1}(M + N(Q_T)) = T^{-1}(M + T(0)) = T^{-1}(M)$. ∎

The next lemma is used in order to show Lemma 3.8.

Lemma 2.9. *Let $T \in CR(X)$ be everywhere defined and let M be a closed subspace of Y such that $T(0) \cap M = \{0\}$ or $T(0) \subset M$. Suppose $M + R(T)$ and $M \cap R(T)$ are closed. If either $T(0)$ or $M \cap R(T)$ has finite dimension then, $R(T)$ is closed.*

Proof. Write for short $N = (M + T(0)) \cap R(T) = M \cap R(T) + T(0)$ if $T(0) \cap M = \{0\}$ and $N = M \cap R(T)$ if $T(0) \subset M$. Clearly N is a closed subspace of X such that $T(0) \subset M$ and hence $T^{-1}(N)$ is closed (by Lemma 2.8). Now we define the linear relation

$$\widehat{T} : (X/T^{-1}(N)) \oplus (M/N) \to (R(T) + M)/N$$

by

$$\widehat{T}(\overline{x} + \overline{m}) := \{\overline{y} + \overline{m} : y \in Tx\}.$$

It is easy to check that $\widehat{T}$ is correctly defined and moreover $\widehat{T}$ is single valued. In fact, for all $y \in T(0)$, $\overline{y} = \overline{0}$ (as $T(0) \subset N$), so that $\widehat{T}(\overline{0}) = \overline{0}$. On the other hand clearly $\widehat{T}$ is surjective and injective. Indeed injective, let $x \in X$ and $m \in M$ such that $\widehat{T}(\overline{x} + \overline{m}) = \overline{0}$ then $Tx + m \subset N$ and hence $Tx \subset (M+N) \cap R(T) = N$. It follows that $m \in N$ and $x \in T^{-1}(N)$, so that $\overline{x} + \overline{m} = \overline{0}$. Since $TT^{-1}(N) = N$ then for all $x \in X, m \in M$ and $y \in Tx$ we have

$$\begin{aligned}
\|\widehat{T}(\overline{x}+\overline{m})\| &= \|\overline{y}+\overline{m}\| = d(y+m,N) \\
&\leq d(y,N) + d(m,N) \\
&= d(y,TT^{-1}(N)) + d(m,N) = \inf_{x' \in T^{-1}(N)} d(y,Tx') + d(m,N).
\end{aligned}$$

Which implies that

$$\begin{aligned}
\|\widehat{T}(\overline{x}+\overline{m})\| &\leq \inf_{x' \in T^{-1}(N)} d(Tx,Tx') + d(m,N) \\
&= \inf_{x' \in T^{-1}(N)} \|Tx - Tx'\| + d(m,N) \\
&\leq \|T\| \inf_{x' \in T^{-1}(N)} \|x - x'\| + d(m,N) \\
&= \|T\| d(x,T^{-1}(N)) + d(m,N) \\
&\leq (1+\|T\|)\big(d(x,T^{-1}(N)) + d(m,N)\big) \\
&= (1+\|T\|)\|\overline{x}+\overline{m}\|.
\end{aligned}$$

Thus $\widehat{T}$ is bounded and since $\widehat{T}$ is bijective then, by the Mapping Theorem of linear operators, $\widehat{T}(X/(T^{-1}(N)))$ is closed. Now let $P : R(T) + M \to (R(T)+M)/N$ be the canonical projection. Then $R(T) = P^{-1}(R(T)/N) = P^{-1}(\widehat{T}(X/T^{-1}(N))$. Thus $R(T)$ is closed. ∎

Lemma 2.10. *Let X, Y be two Banach spaces and $T \in CR(X)$. Then $D(T)$ endowed with the norm $|\cdot|$ defined by*

$$|x| := \|x\| + \inf\{\|y\| : y \in Tx\} = \|x\| + \|Tx\|,$$

is a Banach space.

Proof. Since T is closed then QT is closed (see [9, Proposition II.5.3]), which implies that $G(QT)$ is closed. On the other hand, $T(0)$ is closed (as T is closed) so that $Y/T(0)$ is a Banach space. It follows that $G(QT)$ is complete. Now, clearly $(D(T), |\cdot|)$ and $G(QT)$ are isomorphic. This prove that $(D(T), |\cdot|)$ is complete. ∎

3. Characterization of the descent and the essential descent spectrum

In this section we give characterization and some properties of the descent spectrum and the essential descent spectrum of linear relations everywhere defined in Banach spaces. We investigate that the corresponding spectrums are closed and that the descent (respectively essential descent) of every multivalued linear operator acting in X is finite if and only if X has finite dimension. For this end, we first prove some technical lemmas.

Definition 3.1. Let $T \in LR(X)$, let k and m_i, $1 \leq i \leq n$ be some positive integers, and let $\lambda_i \in \mathbb{C}$, $1 \leq i \leq k$ be some distinct constants. Let $P(X) = \prod_{i=1}^{k}(X - \lambda_i)^{m_i}$. Then the polynomial P in T is the linear relation

$$P(T) := \prod_{i=1}^{k}(T - \lambda_i)^{m_i}.$$

The behaviour of the domain, the range, the null space and the multivalued part of $P(T)$ is described in the following lemma which is due to Sandovici [19].

Lemma 3.1. ([19, 3.2, 3.3, 3.4, 3.5 and 3.6]) *Let T be a linear relation in a vector space X, let $n \in \mathbb{N}, \lambda_i \in \mathbb{K}, m_i \in \mathbb{N}, 1 \leq i \leq k$. Assume that λ_i, $1 \leq i \leq k$ are distinct and let $P(T)$ be as in Definition* 3.1. *Then*

(i) $D(P(T)) = D(T^{\sum_{i=1}^{k} m_i})$.

(ii) $R(P(T)) = \bigcap_{i=1}^{k} R(T - \lambda_i)^{m_i}$.

(iii) $N(P(T)) = \sum_{i=1}^{k} N(T - \lambda_i)^{m_i}$.

(iv) $(T - \lambda)^k(0) = T^k(0)$, *if* $\lambda \in \mathbb{K}$.

(v) $P(T)(0) = T^{\sum_{i=1}^{k} m_i}(0)$.

As consequence of the above lemma we have the next result which will be used in the sequel.

Lemma 3.2. *Let T be a linear relation on X everywhere defined. Let*

$$P(X) = \prod_{i=1}^{k}(X-\lambda_i) = X^k + \sum_{j=0}^{k-1} a_j X^j$$

and

$$Q(X) = \prod_{i=1}^{k}(X-\mu_i) = X^k + \sum_{j=0}^{k-1} b_j X^j.$$

Then

(i) $P(T) = T^k + \sum_{j=0}^{k-1} a_j T^j$.

(ii) $P(T) - Q(T) = (P-Q)(T) + T^k - T^k$.

(iii) $P(T)Q(T) = (PQ)(T)$.

The following purely algebraic lemma helps to read Definition 3.2 below. There exhibits some useful connections between the kernels and the ranges of the iterates T^n of a linear relation T on X.

Lemma 3.3. ([14, Lemma 3.7]) *Let $T \in LR(X)$. Then the following statements are equivalent*

(i) $N(T) \subset R(T^n)$ *for each* $n \in \mathbb{N}$.

(ii) $N(T^m) \subset R(T)$ *for each* $m \in \mathbb{N}$.

(iii) $N(T^m) \subset R(T^n)$ *for each* $m \in \mathbb{N}$ *and* $n \in \mathbb{N}$.

Definition 3.2. We say that a linear relation $T \in LR(X)$ is regular if $R(T)$ is closed and T verifies one of the equivalent conditions of Lemma 3.3.

Trivial examples of regular linear relations are surjective multivalued operators as well as injective multivalued operators with closed range.

Lemma 3.4. *Let $T \in CR(X)$ be regular and everywhere defined with finite codimensional range. Then*

$$\operatorname{codim} R(T^n) = n \ \operatorname{codim} R(T)$$

for all positive integers n.

Proof. Let $n \geq 1$. Since T is regular then, $X/R(T) = X/R(T) + N(T^{n-1})$ and according to Lemma 2.4(i), it follows that $X/R(T) \cong R(T^{n-1})/R(T^n)$. On the other hand, using Lemma 2.2, we obtain that $\dim[(X/R(T^{n-1})) \times (R(T^{n-1})/R(T^n))] = \dim(X/R(T^n))$, which implies that

$$\begin{aligned} \operatorname{codim} R(T^n) = \dim(X/R(T^n)) &= \dim[(X/R(T^{n-1})) \times (R(T^{n-1})/R(T^n))] \\ &= \dim(X/R(T^{n-1})) + \dim(R(T^{n-1})/R(T^n)) \\ &= \operatorname{codim} R(T^{n-1}) + \operatorname{codim} R(T) \end{aligned}$$

Thus, a successive repetition of this argument leads to $\operatorname{codim} R(T^n) = n \operatorname{codim} R(T)$. ∎

As consequence of [8, Theorem 3.1] we have that

Lemma 3.5. *Let $T \in LR(X)$ be everywhere defined and with trivial singular chain. Then*

$$ind(T^n) = n\ ind(T) \tag{3.1}$$

The next lemma is elementary but essential to prove Theorem 3.1.

Lemma 3.6. *Let $T \in LR(X)$ be such that $d_e(T)$ is finite. Then*

$$N(T) \cap R(T^q) = N(T) \cap R(T^{q+n})$$

for some $q \in \mathbb{N}$ and for all $n \in \mathbb{N}$.

Proof. It is very well known (see [16, Lemma 22.2]) that if U, V and W are subspaces of a Banach space X such that $U \subset W$ then,

$$(U + V) \cap W = U + (V \cap W). \tag{3.2}$$

Let $q = q(T)$, and let $\widehat{T} : R(T^q)/R(T^{q+1}) \to R(T^{q+1})/R(T^{q+2})$ be the quotient map defined by $\widehat{T}\overline{x} := \{\widetilde{y} : \ y \in Tx\}$. Then $\widehat{T}\overline{0} = \widetilde{0}$ since $T(0) \subset R(T^{q+2})$. It follows that $\widehat{T}$ is single valued and hence $\widehat{T}\overline{x} = \widetilde{y}$ for any (then for all) $y \in Tx$. According to (3.2), Lemma 2.2 and Lemma 2.1, we obtain that

$$\begin{aligned} \dim N(\widehat{T}) &= \dim[(T^{-1}(R(T^{q+2})) \cap R(T^q))/R(T^{q+1})] \\ &= \dim[(N(T) + R(T^{q+1})) \cap R(T^q)/R(T^{q+1})] \\ &= \dim[((N(T) \cap R(T^q)) + R(T^{q+1}))/R(T^{q+1})] \\ &= \dim[(N(T) \cap R(T^q))/(N(T) \cap R(T^{q+1}))]. \end{aligned}$$

On the other hand it is easy to see that $\widehat{T}$ is surjective. Furthermore, since $\dim R(T^q)/R(T^{q+1}) = \dim R(T^{q+1}/R(T^{q+2})$, then $\widehat{T}$ is injective, which implies that $\dim N(\widehat{T}) = 0$. Therefore, $N(T) \cap R(T^q) = N(T) \cap R(T^{q+1})$. ∎

The next lemma investigates the stability of certain Fredholm type properties of a linear relation under small perturbation. It is used to prove Theorem 3.1.

Lemma 3.7. ([2, Theorem 23, Theorem 25 and Theorem 27]) *Let $T \in CR(X)$ be lower semi-Fredholm and regular. Then there exists $\delta > 0$ such that $T - \lambda$ is both lower semi-Fredholm and regular for all $|\lambda| < \delta$. Moreover $\beta(T - \lambda) = \beta(T)$ for $0 < |\lambda| < \delta$.*

In [6], M. Burgos, A. Kaidi, M. Mbekhta and M. Oudghiri prove that $\rho_{des}(T)$ is an open set for every bounded operator in X. In the following we show that both $\rho_{des}(T)$ and $\rho^e_{des}(T)$ are open for each $T \in CR(X)$ everywhere defined, and if moreover T is regular then $\sigma_{des} \backslash \sigma^e_{des}(T)$ is also open.

Theorem 3.1. *Let $T \in CR(X)$ be everywhere defined for which $d_e(T)$ is finite and suppose $\rho(T) \neq \emptyset$. Then there exists $\delta > 0$ such that for $0 < |\lambda| < \delta$ and $q = q(T)$, we have the following assertions:*

(i) *$T - \lambda$ is regular.*

(ii) *$\operatorname{codim} R(T - \lambda)^n = n \operatorname{codim} R(T) = n \dim(R(T^q)/R(T^{q+1}))$ for all $n \in \mathbb{N}$.*

(iii) *$\dim N(T - \lambda)^n = n \dim(N(T^{q+1})/N(T^q))$ for all $n \in \mathbb{N}$.*

Proof. Since T is closed and $\rho(T) \neq \emptyset$ then T^q is closed [10, Lemma 3.1]. It follows that T^{-q} is closed and hence, from Lemma 2.10, $R(T^q) = D(T^{-q})$ equipped with the norm $|\cdot|$ defined by

$$|y| := \|y\| + \inf\left\{\|x\| : x \in X \text{ and } y \in T^q x\right\} = \|y\| + \|T^{-q}y\|,$$

is a Banach space. Let $T_0 := T_{/R(T^q)}$. Then T_0 is closed and lower semi-Fredholm and hence by [1, Corollary 1.8] $R(T_0)$ is closed. Indeed T_0 is closed, let $(x_n, y_n)_n$ be a sequence of $G(T_0)$ converging to (x, y). Then $x_n \xrightarrow{|\cdot|} x$ and $y_n \xrightarrow{|\cdot|} y$ so that $x, y \in R(T^q)$. Moreover $x_n \xrightarrow{\|\cdot\|} x$ and $y_n \xrightarrow{\|\cdot\|} y$ which implies that $(x, y) \in G(T)$. Thus $(x, y) \in G(T_0)$. Further, since $R(T_0) = R(T^{q+1})$ is of finite codimension in $R(T^q)$ then T_0 is lower semi-Fredholm. According to Lemma 3.6, $N(T_0) = N(T) \cap R(T^q) = N(T) \cap R(T^{q+n}) \subset R(T^{q+n}) = R(T_0^n)$.

Thus T_0 is regular. On the other hand let $\alpha \in \rho(T)$ then $\{0\} = N(T-\alpha) \supset N(T-\alpha) \cap R(T^q) = N(T_0-\alpha)$ and $R(T_0-\alpha) = (T-\alpha)(R(T^q)) = T^q[(T-\alpha)(X)] = T^q(X) = R(T^q)$ (as T^q and $T-\alpha$ commute). This implies that $\alpha \in \rho(T_0)$ and thus $\rho(T_0) \neq \emptyset$. Furthermore, by Lemma 3.7 there exists $\delta > 0$ such that $T_0 - \lambda$ is regular, lower semi-Fredholm and $\beta(T_0-\lambda) = \beta(T_0)$, for all $0 < |\lambda| < \delta$. Now, for $n \geq 1$ and $\lambda \neq 0$, we consider the polynomials P, Q defined by $P(z) = (z-\lambda)^n$ and $Q(z) = z^q$, for all $z \in \mathbb{C}$. Clearly that P and Q have no common divisors, then there exist two polynomials u and v such that $1 = P(z)u(z) + Q(z)v(z)$ for all $z \in \mathbb{C}$. Hence by Lemma 3.2, $I + T^m - T^m = (T-\lambda)^n u(T) + T^q v(T)$ for some $m \geq \max(n,q)$ and thus $X = R(T-\lambda)^n + R(T^q)$. Since T_0 is closed then by Lemma 2.5, $T_0 - \lambda$ is closed. It follows by Lemma 3.4, Lemma 2.2 and Lemma 3.7 that

$$\begin{aligned}
\operatorname{codim} R(T-\lambda)^n &= \dim X/R(T-\lambda)^n \\
&= \dim[(R(T^q) + R(T-\lambda)^n)/R(T-\lambda)^n] \\
&= \dim[R(T^q)/R(T^q) \cap R(T-\lambda)^n] \\
&= \operatorname{codim} R(T_0-\lambda)^n \\
&= n \operatorname{codim} R(T_0-\lambda) \\
&= n \operatorname{codim} R(T_0) \\
&= n \dim R(T^q)/R(T^{q+1}).
\end{aligned}$$

This implies that $T-\lambda$ is semi-Fredholm. Moreover, by Lemma 2.6, $N(T-\lambda) = N(T-\lambda) \cap R(T^q) = N(T_0-\lambda) \subseteq R(T_0-\lambda)^n \subseteq R(T-\lambda)^n$, which implies that $T-\lambda$ is regular. For the statement (iii), since $\rho(T) \neq \emptyset$ then $\rho(T_0) \neq \emptyset$ and hence $\rho(T_0-\lambda) \neq \emptyset$, which implies that $R_c(T_0-\lambda) = \{0\}$ (see Lemma 2.7). Therefore, using (3.1), Lemma 3.7 and Lemma 3.4, we have that

$$\begin{aligned}
\dim N(T-\lambda)^n &= \dim N(T-\lambda)^n \cap R(T^q) \\
&= \dim N(T_0-\lambda)^n \\
&= ind(T_0-\lambda)^n + \operatorname{codim} R(T_0-\lambda)^n \\
&= n[ind(T_0-\lambda) + \operatorname{codim} R(T_0-\lambda)] \\
&= n[ind(T_0) + \operatorname{codim} R(T_0)] \\
&= n \dim N(T_0) = n \dim N(T) \cap R(T^q)
\end{aligned}$$

Since, by Lemma 2.4, $N(T^{q+1})/N(T^q) \cong N(T) \cap R(T^q)$ then

$$\dim N(T-\lambda)^n = n \dim(N(T) \cap R(T^q)) = n \dim(N(T^{q+1})/N(T^q)).$$

∎

As a direct consequence of Theorem 3.1 we obtain the following result for closed multivalued linear operators everywhere defined and with finite ascent.

Corollary 3.1. *Let $T \in CR(X)$ be everywhere defined with finite descent. Suppose $\rho(T) \neq \emptyset$. Then there exists $\delta > 0$ such that for all $0 < |\lambda| < \delta$,*

(i) *$T - \lambda$ is surjective.*

(ii) $\dim N(T-\lambda) = \dim N(T^{q+1})/N(T^q) = \dim(N(T) \cap R(T^q))$ *for some* $q \in \mathbb{N}$.

(iii) *If, moreover, $a(T) < \infty$, then $T - \lambda$ is bijective.*

Corollary 3.2. *Let $T \in CR(X)$ be everywhere defined with $\rho(T) \neq \emptyset$. Let $\lambda_0 \in \sigma(T)$ such that $T - \lambda_0$ has finite ascent and descent. Then λ_0 is an isolated point of $\sigma(T)$ and it is in the boundary of the spectrum of T.*

Proof. Let $\lambda_0 \in \sigma(T)$ such that $T - \lambda_0$ has finite ascent and descent. Clearly by Lemma 2.5 and Lemma 2.7 that $T - \lambda_0$ is closed and $\rho(T - \lambda_0) \neq \emptyset$. Hence there exists by Corollary 3.1, $\delta > 0$ such that $T - \mu$ is injective and surjective for all $0 < |\mu - \lambda_0| < \delta$. This implies that $D(\lambda_0, \delta) \backslash \{\lambda_0\} \subset \rho(T)$. Thus λ_0 is isolated and it is in the boundary of the spectrum of T. ∎

Also as a consequence of Theorem 3.1, we have

Corollary 3.3. *Let $T \in CR(X)$ be everywhere defined and suppose $\rho(T) \neq \emptyset$. Then $\rho_{des}(T)$ and $\rho^e_{des}(T)$ are open subsets and hence $\sigma_{des}(T)$ and $\sigma^e_{des}(T)$ are closed subsets of $\sigma(T)$. Moreover, if T is regular then $\sigma_{des}(T) \backslash \sigma^e_{des}(T)$ is an open set.*

Proof. The openness of $\rho^e_{des}(T)$ and $\rho_{des}(T)$ follows directly from Theorem 3.1 and Corollary 3.1, respectively. Now let $\lambda \in \sigma_{des}(T) \backslash \sigma^e_{des}(T)$ and let $q := q(T - \lambda)$. We see easily, by Lemmas 2.5 and 2.7, that $T - \lambda$ is closed and $\rho(T - \lambda) \neq \emptyset$. Therefore, Theorem 3.1 ensures that there exists an open neighborhood U of λ such that $U \cap \sigma^e_{des}(T) = \emptyset$ and $\operatorname{codim} R(T-\alpha)^n = n \dim(R(T-\lambda)^q / R(T-\lambda)^{q+1})$, for all $\alpha \in U$ and $n \in \mathbb{N}$. Since $T - \lambda$ has infinite descent, $\dim(R(T-\lambda)^q / R(T-\lambda)^{q+1})$ is nonzero, and consequently $(\operatorname{codim} R(T-\alpha)^n)_n$ is a strictly increasing sequence for each $\alpha \in U$. Thus $U \subset \sigma_{des}(T)$, as desired. ∎

For $T \in LR(X)$ we define $E(T) := \rho_{des}(T) \cap \rho_{asc}(T) \backslash \rho(T) = \rho_{des}(T) \cap \rho_{asc}(T) \cap \sigma(T)$. The descent spectrum, and therefore the essential descent spectrum, of a linear relation can be empty. In the next theorem we show that this occurs precisely when $\partial\sigma(T) \subseteq \rho^e_{des}(T)$.

THEOREM 3.2. *Let $T \in CR(X)$ be everywhere defined with $\rho(T) \neq \emptyset$. Then*

$$\rho^e_{des}(T) \cap \partial\sigma(T) = \rho_{des}(T) \cap \partial\sigma(T) = E(T). \tag{3.3}$$

Moreover, the following assertions are equivalent:

(i) $\sigma_{des}(T) = \emptyset$.

(ii) $\sigma^e_{des}(T) = \emptyset$.

(iii) $\partial\sigma(T) \subseteq \rho_{des}(T)$.

(iv) $\partial\sigma(T) \subseteq \rho^e_{des}(T)$.

The proof of this theorem requires the following lemma.

LEMMA 3.8. *Let $T \in CR(X)$ be everywhere defined such that $\rho(T) \neq \emptyset$. Suppose $a(T)$ and $d(T)$ are both finite. Then $R(T^n)$ is closed, for all $n \geq a(T)$.*

Proof. First observe, since T is closed and $\rho(T) \neq \emptyset$ then T^n is closed for all $n \in \mathbb{N}$ (by [10, Lemma 3.1]) and hence $N(T^n)$ is closed. On the other hand $D(T) = X$ leads to $D(T^n) = X$ and since T has finite ascent and descent, it follows by ([18, Theorem 5.7]) that $a(T) = d(T)$. Let $a := a(T), d := d(T)$ and fix $n \geq a$. Lemma 5.5 of [18] leads to $N(T^n) \cap R(T^n) = \{0\}$. Now, since T is everywhere defined, Lemma 5.6 of [18] ensures that $N(T^n) \cap R(T^n) = X$. According to Lemma 2.9, it follows that $R(T^n)$ is closed. ∎

Proof of Theorem 3.2. According to Corollary 3.2, the inclusions $\rho^e_{des}(T) \cap \partial\sigma(T) \supset \rho_{des}(T) \cap \partial\sigma(T) \supset E(T)$ are trivial. Now, let λ in the boundary of the spectrum of T such that $T-\lambda$ has finite essential descent. By Lemma 2.5, $T-\lambda$ is closed and Lemma 2.7 leads to $\rho(T-\lambda) \neq \emptyset$. Then by Theorem 3.1, there exists a punctured neighborhood V of λ such that $\dim N(T-\alpha) = \dim(N(T-\lambda)^p)/(N(T-\lambda)^q)$ and $\operatorname{codim} R(T-\alpha) = \dim(R(T-\lambda)^q/R(T-\lambda)^{q+1})$ for some $q \in \mathbb{N}$ and for all $\alpha \in V$. Moreover $T-\alpha$ is closed for all α (see Lemma 2.5) and, since $\lambda \in \partial\sigma(T)$ and $\rho(T) \neq \emptyset$, then there exists $\alpha_0 \in V \backslash \sigma(T) \neq \emptyset$. Therefore

$$\begin{aligned} 0 &= \dim N(T-\alpha_0) = \operatorname{codim} R(T-\alpha_0) \\ &= \dim(N(T-\lambda)^{q+1})/N((T-\lambda)^q) \\ &= \dim(R((T-\lambda)^q)/R((T-\lambda)^{q+1})). \end{aligned}$$

This means that $T-\lambda$ is of finite ascent and descent. Lemma 3.8 leads to $R(T^{a(T)+1})$ is closed and therefore $\lambda \in E(T)$.

All the desired implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear from (3.3). For (iv) $\Rightarrow$ (i), assume that $\partial\sigma(T) \subseteq \rho^e_{des}(T)$ then $\partial\sigma(T) = E(T)$. According to Lemma 3.2, it follows that all points of $\partial\sigma(T)$ are isolated and hence $\partial\sigma(T) = \sigma(T)$. By (3.3) we conclude that $\sigma(T) \subset \rho_{des}(T) \cap \rho_{asc}(T)$. Which means that $\mathbb{C} = \sigma(T) \cup \rho(T) \subset \rho_{des}(T) \cap \rho_{asc}(T)$. Hence $\rho_{des}(T) = \mathbb{C}$, which means that $\sigma_{des}(T) = \emptyset$. ∎

Corollary 3.4. *Let X be a Banach space and let $K(X) := \{T \in CR(X) : D(T) = X, \text{ and } \rho(T) \neq \emptyset\}$. The following assertions are equivalent:*

(i) *X is finite dimensional.*

(ii) *T is of finite descent for every $T \in K(X)$.*

(iii) *T is of finite essential descent for every $T \in K(X)$.*

(iv) *$\sigma_{des}(T)$ is empty for every $T \in K(X)$.*

(v) *$\sigma^e_{des}(T)$ is empty for every $T \in K(X)$.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are obvious. Using Lemma 2.7 and Lemma 2.5, it is easy to see that $T \in K(X)$ if and only if $T-\lambda \in K(X)$ for all $\lambda \in \mathbb{C}$. It follows by Theorem 3.2, that (iii) entails (ii). For (v) $\Rightarrow$ (i), suppose $\sigma^e_{des}(T)$ is empty for every $T \in K(X)$. Then $\sigma^e_{des}(T)$ is empty for every $T \in \mathcal{L}(X)$ (the space of all bounded operators on X). It follows by [6, Corollary 1.10)] that X is finite dimensional. ∎

Theorem 3.3. *Let $T \in CR(X)$ be everywhere defined such that $\rho(T) \neq \emptyset$ and let Ω be a connected component of $\rho^e_{des}(T)$. Suppose $\rho(T) \neq \emptyset$. Then*

$$\Omega \subset \sigma(T) \quad \text{or} \quad \Omega \backslash E(T) \subseteq \rho(T).$$

Proof. Let $\Omega_r := \{\lambda \in \Omega : T-\lambda \text{ is both regular and lower semi-Fredholm}\}$. According to Theorem 3.1, $\Omega\backslash\Omega_r$ is at most countable, and hence Ω_r is connected. Suppose $\Omega \cap \rho(T)$ is non-empty, then so is $\Omega_r \cap \rho(T)$. Using [4, Corollary 17] and Lemma 3.7 we obtain that $\operatorname{codim} R(T-\lambda) = 0$ and by the continuity of the index (see [4, Theorem 15]), we get that $N(T-\lambda) = 0$ for all $\lambda \in \Omega_r$. Thus $\Omega_r \subseteq \rho(T)$. Consequently, $\Omega\backslash\Omega_r$ consists of isolated points of the spectrum with finite essential descent. That is,

$$\Omega\backslash\Omega_r \subseteq \rho^e_{des}(T) \cap \partial\sigma(T) = E(T).$$

Therefore, $\Omega\backslash E(T) \subseteq \Omega_r \subseteq \rho(T)$. ∎

COROLLARY 3.5. *Let $T \in CR(X)$ be everywhere defined such that $\rho(T) \neq \emptyset$. The following assertions are equivalent:*

(i) *$\sigma(T)$ is at most countable.*

(ii) *$\sigma_{des}(T)$ is at most countable.*

(iii) *$\sigma^e_{des}(T)$ is at most countable.*

In this case, $\sigma^e_{des}(T) = \sigma_{des}(T)$ and $\sigma(T) = \sigma_{des}(T) \cup E(T)$.

Proof. Since the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious, to finish the proof it suffices to show that (iii) $\Rightarrow$ (i). Suppose $\sigma^e_{des}(T)$ is at most countable, then $\rho^e_{des}(T)$ is connected. Since $\rho(T) \neq \emptyset$ it follows, by Theorem 3.3, that $\rho^e_{des}(T)\backslash E(T) \subseteq \rho(T)$. Which implies that $\sigma(T) = \sigma^e_{des}(T) \cup E(T)$ is at most countable. Finally, for the last assertion, assume that $\sigma(T)$ is at most countable. Then by Corollary 3.3, we obtain that $\sigma_{des}(T)\backslash\sigma^e_{des}(T)$ is at most countable and open. Therefore it is empty. Thus $\sigma^e_{des}(T) = \sigma_{des}(T)$. ∎

Remark 3.1. An immediate consequence of the previous corollary that if $T \in CR(X)$ be everywhere defined with $\rho(T) \neq \emptyset$, then $\sigma(T)\backslash\{0\} \subset E(T)$ if and only if $\sigma_{des}(T) \subset \{0\}$ if and only if $\sigma^e_{des}(T) \subset \{0\}$.

4. DESCENT, ESSENTIAL DESCENT SPECTRUM AND PERTURBATIONS

We start this section by the next lemma which are used in the sequel.

LEMMA 4.1. *Let $T \in LR(Y, Z)$ and $S, R \in LR(X)$. If $T(0) \subset N(S)$ or $T(0) \subset N(R)$ then*

$$(R + S)T = RT + ST.$$

Proof. Suppose $T(0) \subset N(R)$, the case $T(0) \subset N(S)$ is similar. The inclusion $(R + S)T \subset RT + ST$ is proved in [9, I.4.2 (d)]. For the reverse inclusion, let $x \in D(RT+ST) = D(RT)\cap D(ST) = T^{-1}(D(R))\cap T^{-1}(D(S))$. Then there exist $y_1 \in Tx\cap D(R)$ and $y_2 \in Tx\cap D(S)$, so that $y_1 - y_2 \in T(0) \subset N(R) \subset D(R)$. Hence $y_2 = (y_2 - y_1) + y_1 \in D(R) \cap D(S) = D(R + S)$. It follows that $Tx \cap D(R+S) \neq \emptyset$. Thus $x \in T^{-1}(D(R+S)) = D(R+S)T$ and hence

$$D(RT + ST) \subset D((R + S)T). \tag{4.1}$$

Now we show that $(RT + ST)x \subset (R + S)Tx$, for $x \in D(RT + ST)$. Let $y \in (RT + ST)x = RTx + STx$. Then $y \in Ry_1 + Sy_2$ for some $y_1, y_2 \in Tx$. Hence $y_1 - y_2 \in T(0) \subset N(R)$. It follows that $y \in R(y_1 - y_2 + y_2) + S(y_2) = R(y_1 - y_2) + R(y_2) + S(y_2) = R(0) + Ry_2 + Sy_2 = Ry_2 + Sy_2 = (R + S)y_2$. Hence $y \in (R + S)Tx$. Thus

$$(RT + ST)x \subset (R + S)Tx \tag{4.2}$$

(4.1) and (4.2) implies that $RT + ST \subset (R + S)T$. The reverse inclusion is proved in [9, I.4.2]. ∎

Lemma 4.2. *Let $T \in CR(X)$ be everywhere defined and F be a bounded operator such that $TF = TF$. Suppose $T(0) \subset N(T)$ or $T(0) \subset N(F)$. Then, for every $n \in \mathbb{N}$,*

(i) $(T + F)^n = \sum_{i=0}^{n} \binom{n}{i} T^{n-i}F^i$, *for all $n \in \mathbb{N}$,*

(ii) $T^n - F^n = (\sum_{i=0}^{n-1} (-1)^i T^{n-1-i}F^i)(T - F)$.

Proof. (i) For $n = 0$ and $n = 1$ the result is trivial. Let $n \geq 1$ and suppose $(T + F)^n = \sum_{i=0}^{n} \binom{n}{i} T^{n-i}F^i$. Using [9, I.4.2 (d) and (e)] and Lemma 4.1, it follows that

$$\begin{aligned}
(T+F)^{n+1} &= (T+F)^n(T+F) = (T+F)^nT + (T+F)^nF \\
&= \left(\sum_{i=0}^{n} \binom{n}{i} T^{n-i}F^i\right)T + \left(\sum_{i=0}^{n} \binom{n}{i} T^{n-i}F^i\right)F \\
&= \sum_{i=0}^{n} \binom{n}{i} T^{n-i+1}F^i + \sum_{i=0}^{n} \binom{n}{i} T^{n-i}F^{i+1} \\
&= \sum_{i=0}^{n} \binom{n}{i} T^{n-i+1}F^i + \sum_{i=1}^{n+1} \binom{n}{i} T^{n-i+1}F^i \\
&= T^{n+1} + \sum_{i=1}^{n} \binom{n+1}{i} T^{n-i+1}F^i + F^{n+1} \\
&= \sum_{i=0}^{n} \binom{n+1}{i} T^{n+1-i}F^i.
\end{aligned}$$

(ii) First observe that $T^n + T^kF^j - T^kF^j = T^n$ for all $0 \leq k \leq n$ and $j \in \mathbb{N}$. It follows that

$$\begin{aligned}
&\Big(\sum_{i=0}^{n-1} T^{n-1-i}F^i\Big)(T-F)\\
&= \big(T^{n-1} + T^{n-2}F + \cdots + F^{n-1}\big)T - \big(T^{n-1} + T^{n-2}F + \cdots + F^{n-1}\big)F\\
&= \big(T^n + T^{n-1}F + \cdots + TF^{n-1}\big) - \big(T^{n-1}F + T^{n-2}F^2 + \cdots + F^n\big)\\
&= T^n - F^n + \big(T^{n-1}F - T^{n-1}F + T^{n-2}F^2 - T^{n-2}F^2 +\\
&\qquad\qquad \cdots + TF^{n-1} - TF^{n-1}\big)\\
&= T^n - F^n.
\end{aligned}$$

∎

LEMMA 4.3. *Let F be an operator and $T \in LR(X)$ be everywhere defined such that $T(0) \subset N(T)$. Suppose that F commutes with T. Then*

$$\dim R(T^{n+k-1})/R(T+F)^n \cap R(T^{n+k-1}) \leq \dim R(F^k), \text{ for all } n, k \geq 1.$$

Proof. Let $y_1, y_2, \ldots, y_m$ be in $R(T^{n+k-1})$ such that $\overline{y_1}, \overline{y_2}, \ldots, \overline{y_m}$ are linearly independent in $R(T^{n+k-1})/R(T^{n+k-1}) \cap R(T+F)^n$. There exist elements $x_1, x_2, \ldots, x_m$ such that $y_i \in T^{n+k-1}x_i$ for $1 \leq i \leq m$. Since F is single-valued and everywhere defined then $(T+F)$ and $(-F)$ commute. We replace T by $T+F$ and F by $-F$ in Lemma 4.2, it follows that $y_i \in (T+F)^n z_i + F^k t_i$, for suitable z_i and t_i. Suppose $m > \dim R(F^k)$, then there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_m$, not all zero such that $\sum_{i=1}^m \alpha_i F^k t_i = 0$. Thus

$$\sum_{i=1}^m \alpha_i y_i \in \sum_{i=1}^m (T+F)^n z_i \subset R(T+F)^n.$$

Furthermore, since the α_i are not all zero, $\overline{y_1}, \overline{y_2}, \ldots, \overline{y_m}$ are not linearly independent in $R(T^{n+k-1})/R(T^{n+k-1}) \cap R(T+F)^n$, a contradiction. Thus $m \leq \dim R(F^k)$. ∎

Now, we are in the position to give the main theorem of this section.

THEOREM 4.1. *Let F be a bounded operator acting in X such that F^k is of finite rank for some nonnegative integer k and let $T \in LR(X)$ be everywhere defined which commutes with F. Suppose $T(0) \subset N(T)$ then*

(i) *$d_e(T)$ is finite if and only if $d_e(T+F)$ is finite.*

(ii) *$d(T)$ is finite if and only if $d(T+F)$ is finite.*

Proof. (i) Since F^k has finite-dimensional rank then, by Lemma 4.3,

$$\dim R(T^{n+k-1})/R(T+F)^n \cap R(T^{n+k-1}) \leq \dim R(F^k) < \infty, \tag{4.3}$$

for all positive integer n. Let $d := d_e(T) < \infty$, then by Lemma 2.3,

$$\dim R(T^d)/R(T^{n+k-1}) < \infty, \quad \text{for all} \quad n \geq d. \tag{4.4}$$

A combination of (4.3) and (4.4) leads to, $\dim R(T^d)/R(T+F)^n \cap R(T^{n+k-1})$ is finite for $n \geq d$. On the other hand $R(T+F)^n \cap R(T^{n+k-1}) \subset R(T+F)^n \cap R(T^d) \subset R(T^d)$. Which implies that $\dim R(T^d)/R(T+F)^n \cap R(T^d)$ is finite and as $\dim R(F^k)$ is finite we obtain that,

$$\dim \big((R(T^d) + R(F^k))/R(T+F)^n \cap R(T^d)\big) < \infty, \quad \text{for all } n \geq d. \tag{4.5}$$

If we replace T by $T+F$ and F by $-F$ in (4.3), we get that

$$\dim R(T+F)^{n+k-1}/R(T^n) \cap R(T+F)^{n+k-1} < \infty.$$

It follows that

$$\dim R(T+F)^{n+k-1}/R(T^d) \cap R(T+F)^{n+k-1} < \infty, \quad \text{for all } n \geq d. \tag{4.6}$$

Combining (4.5) and (4.6) we obtain that

$$\dim \big((R(T^d) + R(F^k))/R(T+F)^n\big) < \infty, \quad \text{for all} \quad n \geq d+k.$$

Therefore

$$\begin{aligned}\dim[R(T+F)^n/R(T+F)^{n+1}] = \dim[(R(T^d) + R(F^k))/R(T+F)^{n+1}] \\ - \dim[(R(T^d) + R(F^k))/R(T+F)^n] < \infty,\end{aligned}$$

for all $n \geq d+k$. Thus $d_e(T+F) < \infty$. For the reverse implication it suffices to replace T by $T+F$ and F by $-F$.

(ii) Let $d = d(T) < \infty$, and for $n \geq d$, let

$$\begin{aligned} a_n(T) :&= \dim R(T^{n+k-1})/R(T+F)^n \cap R(T^{n+k-1}) \\ &= \dim R(T^d)/R(T+F)^n \cap R(T^d), \\ b_n(T) :&= \dim R(T+F)^{n+k-1}/R(T+F)^{n+k-1} \cap R(T^n) \\ &= \dim R(T+F)^{n+k-1}/R(T+F)^{n+k-1} \cap R(T^d). \end{aligned}$$

By Lemma 4.3, $a_n(T) \leq \dim R(F^k) < \infty$. Furthermore clearly $(a_n(T))_n$ is an increasing sequence and hence there exists an integer p such that $a_n(T) = a_p(T)$, for $n \geq p$. It follows that

$$R(T+F)^p \cap R(T^d) = R(T+F)^i \cap R(T^d), \text{ for } i \geq p. \tag{4.7}$$

By interchanging T by $T+F$ and F by $-F$ in Lemma 4.3, we obtain that $b_n(T) \leq \dim R(F^k) < \infty$. Furthermore it is easy to see that $(b_n(T))_n$ is an increasing sequence and consequently there exists $q \geq p$ such that $b_n(T) = b_q(T)$ for $n \geq q$. Now using (4.7) it follows that, for $n \geq q \geq p$,

$$\begin{aligned}\dim R(T^{n+k-1})/R(T+F)^p \cap R(T^d)& \\ &= \dim R(T^{n+k-1})/R(T+F)^{n+k-1} \cap R(T^d)\\ &= \dim R(T^{q+k-1})/R(T+F)^{q+k-1} \cap R(T^d)\\ &= \dim R(T^{q+k-1})/R(T+F)^p \cap R(T^d).\end{aligned}$$

This implies that $R(T+F)^{n+k-1} = R(T+F)^{q+k-1}$, for $n \geq q$. Thus $d(T+F) < q+k < \infty$. ∎

Corollary 4.1. *Let $F \in LR(X)$ be single valued and bounded and let $K_F := \{T \in LR(X) : D(T) = X,\ TF = FT \text{ and } T(0) \subset N(T)\}$. Then the following assertions are equivalent:*

(i) *F^k is of finite rank for some integer k.*

(ii) *$\sigma_{des}(T+F) = \sigma_{des}(T)$ for every $T \in K_F$.*

(iii) *$\sigma^e_{des}(T+F) = \sigma^e_{des}(T)$ for every $T \in K_F$.*

The proof of this corollary requires the following lemma.

Lemma 4.4. ([5, Theorem 3.1] and [6, Theorem 3.1]) *Let $F \in LR(X)$ be a bounded operator. Then the following assertions are equivalent:*

(i) *There exists a positive integer k for which F^k is of finite rank.*

(ii) *$\sigma_{des}(T+F) = \sigma_{des}(T)$ for every bounded operator $T \in LR(X)$ commuting with F.*

(iii) *$\sigma^e_{des}(T+F) = \sigma^e_{des}(T)$ for every bounded operator $T \in LR(X)$ commuting with F.*

Proof of Corollary 4.1. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are an immediate consequences of Theorem 4.1. For the implications (ii) $\Rightarrow$ (i) it suffices to see that $\sigma_{des}(T+F) = \sigma_{des}(T)$ for every $T \in K_F$ implies that $\sigma_{des}(T+F) = \sigma_{des}(T)$ for every bounded operator $T \in LR(X)$ commuting with F and using Lemma 4.4 it follows that (i) holds. Similarly Lemma 4.4 leads to (iii) $\Rightarrow$ (i). This completes the proof. ∎

Also as an immediate consequence of Theorem 4.1 we have that

Corollary 4.2. *Let $T \in LR(X)$ be everywhere defined. Then*

$$\sigma^e_{des}(T) \subset \bigcap_{F \in \mathcal{F}_T(X)} \sigma_{des}(T+F),$$

where $\mathcal{F}_T(X)$ denotes the set of all bounded finite-rank operators F on X commuting with T and such that $T(0) \subset N(T)$.

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